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Standard nearest-neighbour discretizations of Klein–Gordon models cannot preserve both energy and linear momentum

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Abstract

We consider nonlinear Klein–Gordon wave equations and illustrate that standard discretizations thereof (involving nearest neighbours) may preserve either standardly defined linear momentum or standardly defined total energy but not both. This has a variety of intriguing implications for the 'non-potential' discretizations that preserve only the linear momentum, such as the self-accelerating or self-decelerating motion of coherent structures such as discrete kinks in these nonlinear lattices.

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1. Introduction

In the last two decades, the interplay of nonlinearity and spatial discreteness has been increasingly recognized as vital for the understanding of a variety of physical systems [1]. Such contexts range from calcium waves in living cells [2] to the propagation of action potentials through the cardiac tissue [3] and from chains of chemical reactions [4] to applications in superconductivity and Josephson junctions [5], nonlinear optics and fibre/waveguide arrays [6], complex electronic materials [7], Bose–Einstein condensates [8] or the local denaturation of the DNA double strand [9].

On the other hand, spatially discrete systems (of coupled nonlinear ordinary differential equations) are also relevant as discretizations and computational implementations of the corresponding continuum field theories that are applicable to a variety of contexts such as statistical mechanics [10], solid state physics [11], fluid mechanics [12] and particle physics [13] (see also references therein). Nonlinear Klein–Gordon type equations are a prototypical example among such wave models and their variants span a diverse range of applications including Josephson junctions in superconductivity, cosmic domain walls in cosmology, elementary particles in particle physics and denaturation bubbles in the DNA, among others;

see e.g. the review of [14] for the various applications from a nonlinear science viewpoint, [15] for an exposition with a particle physics flavour, and [16] for a recent review of the relevant biophysics applications.

The many-body character of interatomic interactions might suggest that the inclusion of the nearest neighbours in the discretization of the background potential can give a more adequate physical model than that obtained by the classical on-site discretizations. However, even two-body interactions can result in a physically meaningful model including nearest neighbours [17].

Recently, a number of discrete Klein–Gordon equations free of the Peierls–Nabarro potential has been systematically constructed. The Hamiltonian set of such models was obtained by Speight and co-workers [18] using the Bogomol'nyi argument [19]. Then the momentum-conserving discretizations [20] were obtained. In the recent work of [21] it was shown that, in all cases, PNp-free discrete Klein–Gordon models can be formulated by using a two-point discrete version of the first integral of a static continuum Klein–Gordon equation. The latter equation is a two-point nonlinear algebraic equation from which the exact static solutions of the three-point discretizations resulting into PNp-free models can be found (see also [22] where a similar idea was reported for the particular case of the ϕ^4 field). Another Hamiltonian PNp-free model has been very recently discovered by Cooper *et al* [23]. Out of these PNp-free models, it may be worth highlighting [17] for the reason that it illustrates that such models are not meant only as mere discretizations of the continuum model, but may also be physically relevant dynamical systems in their own right. This is an additional reason that establishes the value of examining different discretizations, beyond the obvious one of finding faithful representations of the corresponding continuum limit.

In this communication, we examine some of the key properties that ensue when discretizing nonlinear Klein–Gordon (KG) equations, using nearest-neighbour approximations (which are the most standard ones implemented in the literature; see e.g., [1]). In particular, we focus on the physically relevant invariances of the continuum equation (more specifically, the conservation of the linear momentum and of the total energy of the system) and illustrate the surprising result that if we demand that the energy be conserved, then the momentum cannot be conserved, while if we demand that the momentum be conserved then the energy cannot be conserved (resulting in a so-called non-potential model [24]). Clearly, this is a somewhat counter-intuitive result as one might expect that using techniques from symplectic integration [25] or integrable systems [26], this obstruction may be overcome. However, we note that, to the best of our knowledge there is no general methodology for doing so, and, in fact, many of the relevant KG models such as those involving the ϕ^4 or the Morse potential are not believed to be integrable even in their continuum limit (hence, it would be highly non-trivial, if at all possible, to devise e.g. integrable discretizations thereof).

Our presentation will be structured as follows. First, we will provide the general mathematical setting of KG equations and study their discretizations that conserve linear momentum and energy, comparing and constructing the properties of the two. Then, we are going to give an application of our considerations to the physically relevant ϕ^4 model, i.e., the ubiquitous double well potential. Finally, we will summarize our conclusions and discuss future directions.

2. Setup and analytical results

We consider the Lagrangian of the Klein–Gordon field,

$$L = \int_{-\infty}^{\infty} \left[\frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - V(\phi) \right] \mathrm{d}x, \tag{1}$$

and the corresponding equation of motion,

$$\phi_{tt} = \phi_{xx} - V'(\phi). \tag{2}$$

Assuming that the background potential $V(\phi)$ can be expanded in Taylor series we write

$$V'(\phi) = \sum_{s=0}^{\infty} \sigma_s \phi^s. \tag{3}$$

For brevity, when possible, we will use the notations

$$\phi_{n-1} \equiv l, \qquad \phi_n \equiv m, \qquad \phi_{n+1} \equiv r. \tag{4}$$

We start with a general proof of our main statement, namely that discretizations that preserve linear momentum and energy are mutually exclusive for nearest-neighbour discretizations. As was shown in [20], the standard discretization of equation (2) that preserves the discrete analogue of the linear momentum, defined in a standard way,

$$M = \sum_{n=-\infty}^{\infty} \dot{\phi}_n \left(\phi_{n+1} - \phi_{n-1}\right), \tag{5}$$

is one of the forms

$$\ddot{m} = C\,(l + r - 2m) - \frac{F(r, m) - F(m, l)}{r - l}, \tag{6}$$

where $C = 1/h^2$, where h is the lattice spacing, and the derivative of F is equal to V in the continuum limit ($C \to \infty$). Then,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \sum_n \ddot{\phi}_n(\phi_{n+1} - \phi_{n-1}) = \sum_n [H(\phi_{n+1}, \phi_n) - H(\phi_n, \phi_{n-1})] = 0, \tag{7}$$

where $H(r, m) = C(r^2 + m^2 - 2mr) - F(r, m)$, and the terms $\dot{\phi}_n(\dot{\phi}_{n+1} - \dot{\phi}_{n-1})$ cancel out as a telescopic sum.

However, if the model is potential, for nearest-neighbour discretizations the nonlinear term will be of the form $\tilde{V}(r, m)$ such that the Lagrangian can be written as

$$L = \sum_n \left[\frac{1}{2}\dot{\phi}_n^2 - \frac{C}{2}(\phi_{n+1} - \phi_n)^2 - \tilde{V}(\phi_{n+1}, \phi_n)\right], \tag{8}$$

where the first term gives the kinetic energy, $\mathcal{K}$, and the two other terms give the opposite of the potential energy, $-\mathcal{P}$, so that the total energy is $E = \mathcal{K} + \mathcal{P}$. However, then a model that would enforce both energy and momentum conservation would have to satisfy

$$\frac{F(r, m) - F(m, l)}{r - l} = \frac{\partial}{\partial m}[\tilde{V}(r, m) + \tilde{V}(m, l)]. \tag{9}$$

After multiplying with $r - l$, this, in turn, implies that the cross terms involving all 3 of r, m and l should be presentable in the form

$$r\frac{\partial \tilde{V}(m, l)}{\partial m} - l\frac{\partial \tilde{V}(r, m)}{\partial m} = P(r, m) - P(m, l). \tag{10}$$

This is satisfied only if $\tilde{V}(x, y)$ is a (symmetric) quadratic function in its arguments. However, this is incompatible with the nonlinear nature of the model. Hence, it is not possible to satisfy both conservation laws at once.

Let us now derive the general discrete Klein–Gordon model of the form of equation (6) conserving momentum. For the polynomial background forces equation (3), the nonlinear

term of equation (6) can be presented as the sum of s-order terms

$$B(l,m,r) = \frac{F(r,m) - F(m,l)}{r-l} = \sum_{s=0}^{\infty} B_s(l,m,r), \tag{11}$$

with

$$B_s = \sum_{i=0}^{s}\sum_{j=i}^{s} b_{ij,s} r^i m^{j-i} l^{s-j}, \tag{12}$$

where

$$\sum_{i=0}^{s}\sum_{j=i}^{s} b_{ij,s} = \sigma_s, \tag{13}$$

is the continuity constraint.

In the continuum limit one has $l \to m$ and $r \to m$ and thus, equation (11) together with equation (13) ensure the desired limit, $V'(\phi)$. Furthermore, equation (12) takes into account all possible combinations of powers of l, m and r. Coefficients $b_{ij,s}$ make a triangular matrix of size $(s+1) \times (s+1)$. Let us find the coefficients $b_{ij,s}$ to satisfy equation (11). We write

$$(r-l)B_s = \sum_{i=0}^{s}\sum_{j=i}^{s} b_{ij,s} r^{i+1} m^{j-i} l^{s-j} - \sum_{i=0}^{s}\sum_{j=i}^{s} b_{ij,s} r^i m^{j-i} l^{s-j+1}. \tag{14}$$

Terms containing both l and r should be cancelled out because they do not fit the representation of equation (11). This can be achieved by setting $b_{ij,s} = b_{(i+1)(j+1),s}$, i.e., coefficients in each diagonal of the triangular matrix must be equal. The simplified expression reads

$$(r-l)B_s = \sum_{i=0}^{s} b_{is,s} r^{i+1} m^{s-i} - \sum_{i=0}^{s} b_{0i,s} m^i l^{s-i+1}. \tag{15}$$

To symmetrize the result, we add and subtract $b_{00,s} m^{s+1}$

$$\begin{aligned}(r-l)B_s = {} & b_{00,s}(r^{s+1} + m^{s+1}) - b_{00,s}(m^{s+1} + l^{s+1}) \\ & + \sum_{i=1}^{s} b_{0(s-i+1),s} r^i m^{s-i+1} - \sum_{i=1}^{s} b_{0i,s} m^i l^{s-i+1},\end{aligned} \tag{16}$$

where we shifted the summation index by 1 in the first sum and also used the equality of the diagonal coefficients. The desired representation is obtained for arbitrary $b_{00,s}$ and arbitrary $b_{0i,s} = b_{0(s-i+1),s}$ for $i > 0$. Summing up, (i) the coefficients $b_{ij,s}$ within each diagonal are equal, (ii) the coefficients on the main diagonal can be chosen arbitrarily, and (iii) the terms on ith super-diagonal ($i > 0$) must have the same coefficients as the terms on $(s-i+1)$th diagonal (and these can also be chosen arbitrarily).

For B_s the number of super-diagonals is s so that the number of free coefficients is $1 + \lceil s/2 \rceil$, where $\lceil x \rceil$ is lowest integer greater than or equal to x. We must also take into account the continuity constraint of equation (13) and the number of free coefficients becomes $\lceil s/2 \rceil$.

For example, the coefficients of B_3 are

$$b_{ij,3} = \begin{bmatrix} b_{00,3} & b_{01,3} & b_{02,3} & b_{01,3} \\ & b_{00,3} & b_{01,3} & b_{02,3} \\ & & b_{00,3} & b_{01,3} \\ & & & b_{00,3} \end{bmatrix}, \tag{17}$$

$$4b_{00,3} + 4b_{01,3} + 2b_{02,3} = \sigma_3.$$

Since the model equation (6) is translationally invariant, the static kink is free of the Peierls–Nabarro potential (PNp) [20], i.e., the periodic potential that nonlinear waves have to overcome to move by one lattice site [14] (see also references therein). This is an important qualitative difference with respect to the conventional discretization when, in equation (1), $V(\phi)$ is substituted with $V(\phi_n)$ and thus, in the equation of motion equation (2), $V'(\phi)$ becomes $V'(\phi_n)$.

Another class of Klein–Gordon models which support energy conservation and sustain static kinks but which are free of PNp has been derived by Speight and collaborators [18]. In such models the background potential term of equation (1), $P = \int V(\phi)\,\mathrm{d}x$, should be discretized as

$$P = \sum_n \left(\frac{G(\phi_{n+1}) - G(\phi_n)}{\phi_{n+1} - \phi_n} \right)^2, \qquad \text{with} \quad [G'(\phi)]^2 = V(\phi). \tag{18}$$

3. Numerical results/model comparison

We now examine various models proposed as discretizations of the continuum field theory in the context of perhaps one of the most famous such examples, namely the double-well ϕ^4 model [10, 11, 13] (see also the review [15]).

The discrete Klein–Gordon model conserving momentum is given by equation (6) with the nonlinear term of equation (11) where the coefficients $b_{ij,s}$ are as described in the previous section. The continuum ϕ^4 model has the background potential $V(\phi) = (1-\phi^2)^2/4$, hence $V'(\phi) = -\phi + \phi^3$ so that in equation (3) all $\sigma_s = 0$ except for $\sigma_1 = -\sigma_3 = -1$. The momentum-preserving discretization then reads

$$\begin{aligned}\ddot{m} = {} & (C+\alpha)(l+r-2m) + m - \beta(l^2+lr+r^2) + \beta m(l+r+m) - \gamma(l^3+r^3+l^2r+lr^2) \\ & - \delta m(l^2+m^2+r^2+lr) - \tfrac{1}{2}(1-4\gamma-4\delta)m^2(l+r),\end{aligned} \tag{19}$$

where $\alpha = -b_{00,1}$, $\beta = b_{00,2}$, $\gamma = b_{00,3}$, $\delta = b_{01,3}$ are free parameters and we did not include the terms with $s > 3$.

The model of equation (19) will be compared to the model obtained from equation (18) in ϕ^4 case [18], namely

$$\ddot{m} = \left(C + \tfrac{1}{6}\right)(l+r-2m) + m - \tfrac{1}{18}[2m^3 + (m+l)^3 + (m+r)^3], \tag{20}$$

and also to the 'standard' ϕ^4 discretization, i.e.,

$$\ddot{m} = C(l+r-2m) + m - m^3. \tag{21}$$

If in equation (19), $\alpha = \beta = \gamma = \delta = 0$, then the models of equation (19) and equation (20) have the same linear vibration spectrum (i.e., dispersion relation $\omega = \omega(\kappa)$) for the vacuum solution $\phi_n = \pm 1$, namely $\omega^2 = 2 + (4C-2)\sin^2(\kappa/2)$. This can be compared to the spectrum of the vacuum of equation (21), $\omega^2 = 2 + 4C\sin^2(\kappa/2)$.

We analyse the kink internal modes (i.e., internal degrees of freedom [27]) for these three models. First, we determine the kink-like heteroclinic solution by means of relaxational dynamics. Then, the linearized equations are used in a lattice of $N = 200$ sites to obtain N eigenfrequencies and the corresponding eigenmodes. We are particularly interested in the eigenfrequencies which lie outside the linear vibration band of vacuum solution and thus are associated with the kink internal modes. It is worthwhile to note that the eigenproblem for models conserving energy, equations (20) and (21), has a symmetric Hessian matrix while the non-self-adjoint problem for the momentum conserving model equation (19) results in a *non-symmetric* matrix.

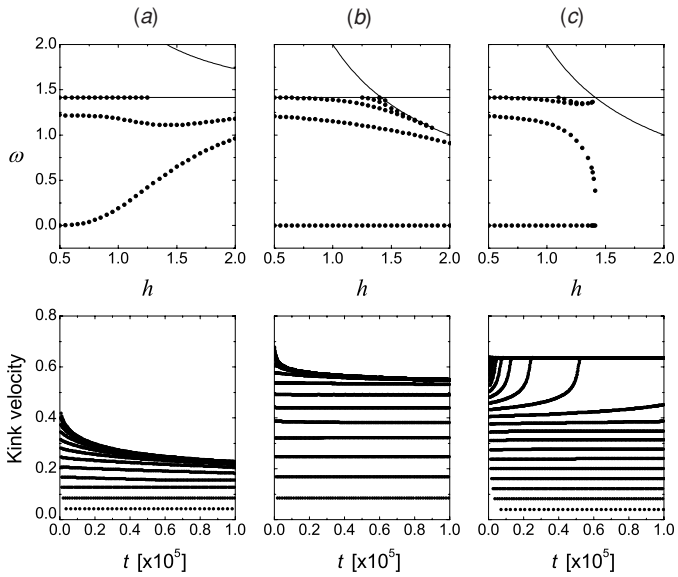

Figure 1. Upper panels: boundaries of the linear spectrum of the vacuum (solid lines) and kink internal mode frequencies (dots) as functions of the lattice spacing $h = 1/\sqrt{C}$. Lower panels: time evolution of kink velocity for different initial velocities and $h = 0.7$. The results are shown for (*a*) classical ϕ^4 model, equation (21), (*b*) PNp-free model conserving energy, equation (20), and (*c*) PNp-free model conserving momentum, equation (19), with $\alpha = \beta = \gamma = \delta = 0$.

The top panels of figure 1 present the boundaries of the linear vibration spectrum of the vacuum (solid lines) and the kink internal modes (dots) as the functions of lattice spacing h for (a) the classical ϕ^4 model of equation (21), (b) the PNp-free model of equation (20) conserving energy, and (c) the PNp-free model of equation (19) conserving momentum. In PNp-free models kinks possess a zero frequency, Goldstone translational mode similarly to the continuum ϕ^4 kink. Hence, the static kink can be centred anywhere on the lattice. The results presented in figure 1 are for the kink situated exactly between two lattice sites. This position is the stable position for the classical ϕ^4 discrete kink [27]. Since all three discrete models share the same continuum (ϕ^4) limit, their spectra are very close for small $h(<0.5)$. We found that the model equation (19) may have kink internal modes lying *above* the spectrum of vacuum, e.g., for $\alpha = 1/2$, $\beta = 0$, $\gamma = 1/4$ and $\delta = 0$. Such modes are short-wavelength ones, with large amplitudes (energies) and they do not radiate because of the absence of coupling to the linear phonon spectrum.

Perhaps more interesting are the implications of such discretizations on the mobility of these localized coherent structures. In the PNp-free models, equations (19) and (20), the kink was launched using a perturbation along the Goldstone mode to provide the initial kick. In the classical model equation (21) for this purpose we employed the imaginary frequency (real eigenvalue) unstable eigenmode for a kink initialized at the unstable position (a 'site-centred' kink). In all cases the amplitude of the mode is related to the initial velocity of the kink. In the bottom panels of figure 1 we present the time evolution of the kink velocity for different initial velocities and $h = 0.7$ for the three discretizations. The results suggest that the mobility of the kink in the classical ϕ^4 model presented in (*a*) is much smaller than in the PNp-free models, (*b*) and (*c*). Furthermore, a very interesting effect of kink *self-acceleration* can be observed in panel (*c*). Here there exists a selected kink velocity $v^* \approx 0.637$ and kinks launched with

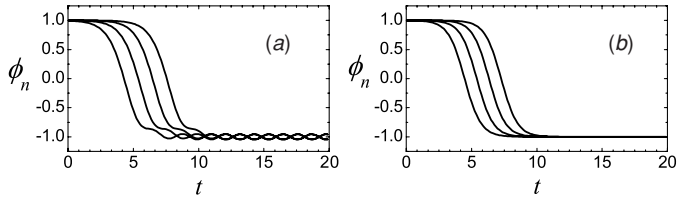

Figure 2. Trajectories of particles (a) in the model of equation (19) with $h = 0.7$ when the kink moves with a steady velocity v^* (see figure 1(c), bottom panel) and (b) for the continuum ϕ^4 kink.

$v > v^*$, in a very short time adjust their velocities to v^*. More surprisingly, the velocity adjustment is observed even for kinks launched with $v < v^*$. In the steady-state regime, when the kink moves with $v = v^*$, it excites (in its tail) the short-wave oscillatory mode even though in front of the kink the vacuum is not perturbed.

These results generate the question of where the energy for the self-acceleration and vacuum excitation comes from. In figure 2(a) we show the trajectories of four neighbouring particles when a kink moving with $v = v^*$ (see figure 1(c), bottom panel) moves through. For comparison, in (b) the trajectories for the classical ϕ^4 kink,

$$\phi_n(t) = \tanh[\rho(nh - vt)], \tag{22}$$

where $\rho = [2 - 2v^2]^{-1/2}$, are shown. In both cases the trajectories are identical and shifted with respect to each other by $t = h/v$, but in (b) they are the odd functions with respect the point $\phi_n = 0$ while in (a) they are not. The work done by the background forces, equation (11), to move the nth particle from one energy well to another is $W_n = -\int_{-\infty}^{\infty} \dot{m} B(l, m, r)\, \mathrm{d}t$. For the ϕ^4 model equation (19) with $\beta = \gamma = \delta = 0$, the nonlinear part of $B(l, m, r)$ reduces to $B(l, m, r) = (1/2)m^2(l + r)$. Obviously, $W_n = 0$ for any trajectory for the classical ϕ^4 kink. It is straightforward to demonstrate that $W_n = 0$ for the kink in the momentum-conserving model for a trajectory having the odd symmetry, e.g., equation (22). However, if a term breaking the odd symmetry, e.g., $\varepsilon \cosh^{-1}[\theta(nh - vt)]$, is added to equation (22), the work becomes nonzero, $W_n = \frac{\pi}{2}\varepsilon(\varepsilon^2 + 1)[\cosh(\rho h) - 1]^3/\sinh^4(\rho h)$, where we set for simplicity $\theta = \rho$. Numerically we found that W_n can be positive or negative depending on ρ, θ and the kink velocity, v. This simple analysis qualitatively explains the kink self-acceleration or deceleration and the vacuum excitation. The energy for this comes from the breaking of the odd symmetry of particle trajectories, which is possible in the case of path-dependent background forces. It is, thus, very interesting to highlight the distinctions between the 'regular' discrete models, the PNp-free, energy conserving discrete models and the PNp-free, momentum conserving discrete models. The first ones lead to rapid dissipation of the wave's kinetic energy due to the PN barrier. The second render the dissipation far slower in time. Finally the third may even sustain self-accelerating waves and locking to a particular speed due to the non-potential nature of the relevant model.

Let us describe one physical phenomenon, namely, flutter, where the self-acceleration effect, similar to that observed in our simulations, can be observed. Consider an elastic body (such as a wing) placed in a flow of gas. In stable regimes, the wing does not accept the energy from the gas flow. For some gas velocities, the flutter instability can occur, resulting in the appearance of vibration of the wing with an amplitude growing in time. In this regime the wing obtains the energy from the gas flow. If the wing is strong enough to sustain high-amplitude vibrations, this regime can be stabilized by the nonlinearity in the elastic response of the wing and by increased energy dissipation. This picture is in good qualitative agreement with kink dynamics presented in the bottom panel of our figure 1(c). Slow kinks move with

constant velocity with no energy exchange with the surroundings. Faster kinks show the self-acceleration effect due to the energy exchange with the surroundings. Simultaneously, the energy lost for vacuum excitation increases (see figure 2(*a*)) and finally the motion is stabilized.

4. Conclusions and future challenges

The statement that Klein–Gordon discrete model cannot conserve energy and momentum simultaneously was proved for the case of standard nearest-neighbour discretizations. This raises the issue of how to gauge the 'adequacy' of a discretization scheme with respect to the continuum model dynamics. This is also important because it offers significant new insights on what type of results one should be expecting for different types of discretizations as regards their dynamical behaviour and how these should be expected to differ from the corresponding continuum limit or from those of other discretizations. In view of this question, a number of characteristic similarities and differences between energy- and momentum-conserving discrete models were highlighted. The momentum conserving Klein–Gordon system with non-potential background forces discussed here differs from other path-dependent systems, e.g., having friction and/or ac drive, in the sense that the viscosity and external forces are not explicitly introduced. This makes the dynamics of the system somewhat 'peculiar' since, for instance, as it was demonstrated, the existence, the intensity and the sign of energy exchange with the surroundings depends on the symmetry and other characteristics of the motion. Dynamics in non-Hamiltonian systems are less canonical and their analysis becomes more complicated but this does not mean that they are less physically important. In fact, in actual practice, any real physical system is an open system. Sometimes the energy exchange with the surroundings can be neglected, but sometimes it plays a crucial role, as, e.g., in the case of flutter mentioned above which bears interesting qualitative resemblances to the kink dynamics observed herein.

Another area where the faithfulness of the relevant discretizations to their continuum limits can be seriously tested is the outcome of solitary wave collisions. This is a rather stringent test, especially given the sensitivity/complexity of the corresponding phenomenology [13, 15] (see also [28]). In fact, it has already been argued that occasionally some of the PNp-free models are not necessarily good representations of the continuum dynamical behaviour unless the lattice spacing is very small [29]. A more general and systematic comparison of this aspect of different discretizations is still lacking and is an especially interesting problem for future study. Our preliminary results in this direction are in line with the sensitivity across discretizations observed in [29] and seem to indicate that only sufficiently near the limit of $h \to 0$ (typically for $h < 0.1$ or so), one should expect to observe the same phenomenology across discretizations. The careful and detailed analysis of this problem is deferred to a future study.

In the same context (of collisions), however, let us add a few relevant notes. The number of conserved quantities is very important for the prediction of the outcome of solitary wave collisions for nearly integrable models, e.g., weakly discrete versions of the integrable continuum equations, such as the sine-Gordon equation, $u_{tt} - u_{xx} = \sin u$, [30]. In this case, if the number of degrees of freedom of the colliding solitons is greater than the number of quantities conserved with high accuracy, then the near-separatrix energy exchange is possible for particular collision phases [31]. The intensity of near-separatrix energy exchange is proportional to h, while the other manifestations of inelasticity of collision, such as emission of radiation and excitation of soliton's internal modes, are proportional to h^2 and thus, can be neglected for small h. A sine-Gordon kink has one free parameter (velocity) while in the

regime of weak discreteness the sine-Gordon equation conserves energy and momentum (one of these quantities exactly and another one with a high accuracy), and the near-separatrix energy exchange is possible only in three-soliton collisions, e.g., in collisions between a kink and a breather. As for the strongly discrete systems, collisions can be purely elastic only under condition of complete integrability. However, for the Klein–Gordon equation in the form of equation (2), integrable discretizations are not known. We also do not know a systematic study of the influence of the number and type of conserved quantities on the inelasticity of soliton collisions in the highly-discrete systems. In the case of high discreteness, the absence or presence of the Peierls–Nabarro potential can be very important not only with respect to the propagation ability of solitons but also for the outcome of their collisions. Collisions in the non-Hamiltonian systems, for solitons in the regime of steady motion with constant speed, at the best of our knowledge, is a completely unexplored area.

One natural question that the astute reader may pose is how to overcome the limitations imposed on the choice of discretization by our result. While we do not provide a complete answer to this admittedly difficult, as much as it is important, question, we offer the following comments. The response depends to some extent on the scope of the study. If the scope is to represent the continuum limit within a desired accuracy, then one can use lattice models with spacings small enough that will enforce the accuracy sought. On the other hand, if the purpose is to devise a discretization that does not have this problem (or the problem is too computationally expensive to tackle with the above-implied choice of h), then we do not have an answer to offer beyond saying that one should search for such discretizations outside the conditions of the present theorem. For example, one can try to include next-nearest neighbours in the discretization, or to obtain a non-standard momentum definition/conservation law with the right continuum limit.

This illustrates the need for further investigation of the intriguing dynamic properties of such non-potential models in both lattice and continuum settings, especially given the relevance of such path-dependent forces in various applications such as, e.g., aerodynamic and hydrodynamic forces, the forces induced in automatic control systems and others. Such studies are in progress and will be reported in future publications.

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